1 a AB is not defined since the product of a 2×2 and a 3×1 matrix is not defined.
AC is defined since the product of a 2×2 and a 2×1 matrix is a 2×1 matrix
CD is defined since the product of a 2×1 and a 1×2 matrix is a 2×2 matrix
BE is defined since the product of a 1×3 and a 3×1 matrix is a 1×1 matrix

b $DA = \begin{bmatrix} -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -12 \end{bmatrix}$

$$\begin{aligned} A^{-1} &= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{1(-1)-(4)(2)} \begin{bmatrix} -1 & -4 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{-9} \begin{bmatrix} -1 & -4 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{9} & \frac{4}{9} \\ \frac{2}{9} & -\frac{1}{9} \end{bmatrix} \end{aligned}$$

2 a $$\begin{aligned} (A+B)(A-B) &= \begin{bmatrix} -1 & 1 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 4 \\ 18 & -24 \end{bmatrix}. \end{aligned}$$

b Firstly,

$$A^2 = \begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -10 \\ 10 & 12 \end{bmatrix},$$

$$B^2 = \begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 3 & 28 \end{bmatrix},$$

Therefore,

$$\begin{aligned} A^2 - B^2 &= \begin{bmatrix} -3 & -10 \\ 10 & 12 \end{bmatrix} - \begin{bmatrix} 7 & 9 \\ 3 & 28 \end{bmatrix} \\ &= \begin{bmatrix} -10 & -19 \\ 7 & -16 \end{bmatrix}. \end{aligned}$$

3 The matrix is not invertible if and only if its determinant is zero.

$$\begin{aligned} \det \begin{bmatrix} 1 & 2 \\ 4 & x \end{bmatrix} &= 0 \\ 1 \times x - 2 \times 4 &= 0 \\ x - 8 &= 0 \\ x &= 8 \end{aligned}$$

4 Suppose that $A = \begin{bmatrix} x \\ y \end{bmatrix}$. This matrix equation is equivalent to the pair of equations,

$$\begin{aligned} 3x - y &= 5, \quad (1) \\ -6x + 2y &= 10. \quad (2) \end{aligned}$$

Notice that equation (1) is equivalent to equation (2). Therefore, we really have one equation,

$$3x - y = 5.$$

There are infinitely many solutions to this equation. Let $x = t \in \mathbb{R}$. Then

$$y = 3x - 5 = 3t - 5.$$

Therefore

$$\mathbf{A} = \begin{bmatrix} t \\ 3t - 5 \end{bmatrix}.$$

5 $\mathbf{AB} = \begin{bmatrix} -1 & -2 & 3 \\ -5 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -1 & -6 \\ -3 & -8 \end{bmatrix} = \begin{bmatrix} -9 & -8 \\ -15 & 10 \end{bmatrix}$

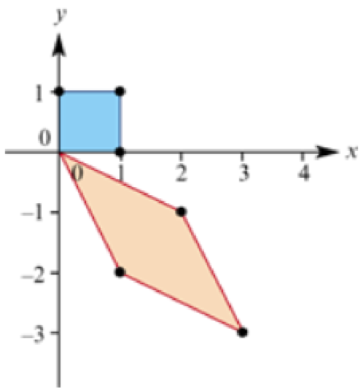
$$\begin{aligned} \mathbf{C}^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{(-1)(-4)-(2)(3)}\begin{bmatrix} -4 & -2 \\ -3 & -1 \end{bmatrix} \\ &= \frac{1}{-2}\begin{bmatrix} -4 & -2 \\ -3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix} \end{aligned}$$

6 a $(2,3) \to (2\times 2+3, -2-2\times 3) = (7,-8)$

b The entries of the matrix are the coefficients of x and y in the transformation,

$$B = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}.$$

c



The area of the original square is 1. Therefore, the image will have area,

$$\begin{aligned} \text{area of image} &= |\det B| \times \text{original area} \\ &= |2(-2)-(1)(-1)| \times 1 \\ &= 3. \end{aligned}$$

d The inverse transformation will have matrix

$$\begin{aligned} B^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{2(-2)-(1)(-1)}\begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix} \\ &= \frac{1}{-3}\begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{bmatrix} \end{aligned}$$

Therefore, the rule for the inverse transformation is $(x,y) \to (\frac{2}{3}x + \frac{1}{3}y, -\frac{1}{3}x - \frac{2}{3}y)$

7 a $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

b $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

c $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

d $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

e $\begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix}$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

f $\begin{bmatrix} \cos(-30^\circ) & -\sin(-30^\circ) \\ \sin(-30^\circ) & \cos(-30^\circ) \end{bmatrix}$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

g $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

h $\begin{bmatrix} \cos(60^\circ) & \sin(60^\circ) \\ \sin(60^\circ) & -\cos(60^\circ) \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

8 a Since $\tan\theta = 4 = \frac{4}{1}$,

we draw a right angled triangle with opposite and adjacent lengths 4 and 1 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{17}$. Therefore

$$\cos\theta = \frac{1}{\sqrt{17}} \text{ and } \sin\theta = \frac{4}{\sqrt{17}}.$$

We then use the double angle formulas to show that

$$\begin{aligned} \cos 2\theta &= 2\cos^2\theta - 1 \\ &= 2\left(\frac{1}{\sqrt{17}}\right)^2 - 1 \\ &= \frac{2}{17} - 1 \\ &= -\frac{15}{17}, \\ \sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\frac{4}{\sqrt{17}}\frac{1}{\sqrt{17}} \\ &= \frac{8}{17}. \end{aligned}$$

Therefore, the required matrix is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} -\frac{15}{17} & \frac{8}{17} \\ \frac{8}{17} & \frac{15}{17} \end{bmatrix}.$$

b $\begin{bmatrix} -\frac{15}{17} & \frac{8}{17} \\ \frac{8}{17} & \frac{15}{17} \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{2}{17} \\ \frac{76}{17} \end{bmatrix}$

Therefore, the image is the point $\left(\frac{2}{17}, \frac{76}{17}\right)$.

9 a The matrix that will reflect the plane in the y-axis is given by

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix that will dilate the result by a factor of 2 from the x-axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

b The matrix that will rotate the plane by $90°$ anticlockwise is given by

$$\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The matrix that will reflect the result in the line $y = x$ is given by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

c The matrix that will reflect the plane in the line $y = -x$ is given by

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

The matrix that will shear the result by a factor of 2 in the x-direction is

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}.$$

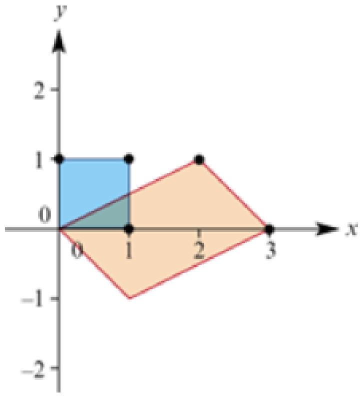
10a
$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -x+2 \\ y-1 \end{bmatrix} \end{aligned}$$

Therefore, the transformation is $(x, y) \to (-x+2, y-1)$.

b
$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right) \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x+2 \\ y-1 \end{bmatrix} \\ &= \begin{bmatrix} -x-2 \\ y-1 \end{bmatrix} \end{aligned}$$

Therefore, the transformation is $(x, y) \to (-x - 2, y - 1)$.

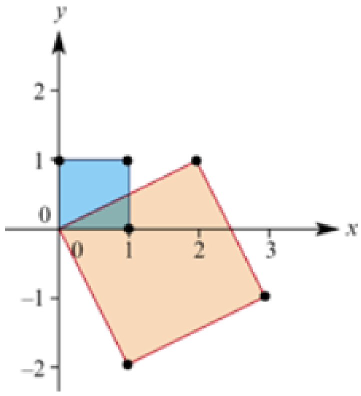
11a



The area of the original square is 1. Therefore, the image will have area,

$$\begin{aligned}\text{area of image} &= |\det B| \times \text{original area} \\ &= |1(1) - (2)(-1)| \times 1 \\ &= 3.\end{aligned}$$

b



The area of the original square is 1. Therefore, the image will have area,

$$\begin{aligned}\text{area of image} &= |\det B| \times \text{original area} \\ &= |2(-2) - (1)(1)| \times 1 \\ &= 5.\end{aligned}$$

12a Firstly, the matrix that will reflect the plane in the line $y = x$ is given by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore the required transformation is

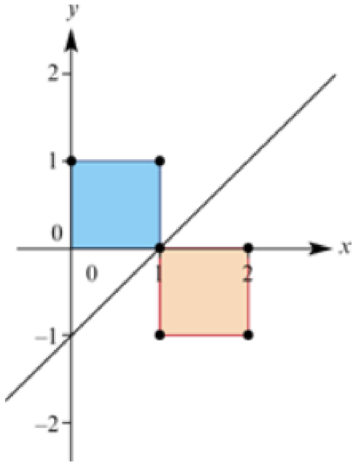
$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y+1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} y+1 \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} y+1 \\ x-1 \end{bmatrix}.\end{aligned}$$

b To find the image of $(0, 0)$ we let $x = 0$ and $y = 0$ so that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0+1 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Therefore, $(0, 0) \to (1, 1)$, as expected.

c



13a $\boldsymbol{a}\cdot\boldsymbol{a} = (2)(2) + (-3)(-3) = 4 + 9 = 13$

b $\boldsymbol{b}\cdot\boldsymbol{b} = (-2)(-2) + (3)(3) = 4 + 9 = 13$

c $\boldsymbol{a}\cdot\boldsymbol{a} = (-3)(-3) + (-2)(-2) = 4 + 9 = 13.$

d $\boldsymbol{a}\cdot\boldsymbol{b} = (2)(-2) + (-3)(3) = -4 - 9 = -13.$

e

$$\begin{aligned}\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) &= (2\boldsymbol{i} - 3\boldsymbol{j})\cdot(-5\boldsymbol{i} - 5\boldsymbol{j})\\ &= (2)(-5) + (-3)(-5)\\ &= 5\end{aligned}$$

f

$$\begin{aligned}(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}+\boldsymbol{c}) &= \boldsymbol{0}\cdot(-\boldsymbol{i} - 5\boldsymbol{j})\\ &= 0\end{aligned}$$

g

$$\begin{aligned}(\boldsymbol{a}+2\boldsymbol{b})\cdot(3\boldsymbol{c}-\boldsymbol{b}) &= (-2\boldsymbol{i} + 3\boldsymbol{j})\cdot(-7\boldsymbol{i} - 9\boldsymbol{j})\\ &= (-2)(-7) + (3)(-9)\\ &= -13\end{aligned}$$

14a

$$\begin{aligned}m\overrightarrow{OA} + n\overrightarrow{BC} &= 2\boldsymbol{i} + 10\boldsymbol{j}\\ m(4\boldsymbol{i} + 2\boldsymbol{j}) + n(9\boldsymbol{i} - \boldsymbol{j}) &= 2\boldsymbol{i} + 10\boldsymbol{j}\\ (4m\boldsymbol{i} + 2m\boldsymbol{j}) + (9n\boldsymbol{i} - n\boldsymbol{j}) &= 2\boldsymbol{i} + 10\boldsymbol{j}\\ (4m + 9n)\boldsymbol{i} + (2m - n)\boldsymbol{j} &= 2\boldsymbol{i} + 10\boldsymbol{j}\end{aligned}$$

Therefore

$$4m + 9n = 2 \text{ and } 2m - n = 10.$$

These simultaneous equations have solution

$$m = \frac{46}{11} \text{ and } n = -\frac{18}{11}.$$

b Since

$$\overrightarrow{OB} = -\boldsymbol{i} + 7\boldsymbol{j},$$

$$\overrightarrow{CD} = (p-8)\boldsymbol{i} - 8\boldsymbol{j},$$

We have,

$$\begin{aligned}\overrightarrow{OB}\cdot\overrightarrow{CD} &= 0\\ (-\boldsymbol{i} + 7\boldsymbol{j})\cdot((p-8)\boldsymbol{i} - 8\boldsymbol{j}) &= 0\\ (-1)(p-8) + (7)(-8) &= 0\\ -p + 8 - 56 &= 0\\ p &= -48.\end{aligned}$$

c Since

$$\overrightarrow{AD} = (p-4)\boldsymbol{i} - 4\boldsymbol{j},$$

we have,

$$\begin{aligned}
|\overrightarrow{AD}| &= \sqrt{17} \\
\sqrt{(p-4)^2 + (-4)^2} &= \sqrt{17} \\
(p-4)^2 + 16 &= 17 \\
(p-4)^2 &= 1 \\
p - 4 &= \pm 1 \\
p &= 3, 5.
\end{aligned}$$